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G-model sets and their self-similarities

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Abstract. The *model sets* (also called *cut and project sets*), first defined by Yves Meyer in harmonic analysis, play a central role in quasicrystal modelling. Each of them is defined by using a *cut and project scheme* containing two projectors and a lattice. We present a method which can be used to study the *self-similarities* of a model set based on the matrices of these projectors in a basis of the lattice. This method also allows one to study the self-similarities of the diffraction spectrum of a model set because, generally, the Bragg peaks with intensity above a given threshold also form a model set. The diffraction pattern corresponding to a quasicrystal is invariant under a finite group G, and the local structure of the quasicrystal can be described by using a finite union of orbits of G, called a G *cluster*. The neighbours of each atom belong to some orbits of G, and the quasicrystal can be regarded as a union of interpenetrating partially occupied translations of the corresponding G cluster. We present a method to obtain a model set (called the G*-model set*) by starting from a G cluster. The experimental diffraction patterns allow one to determine the symmetry group G, and high-resolution electron microscopy images enable one to choose a G cluster describing the local structure. The existing computer programs for the cut and project method allow one to pass directly from the local structure of the quasicrystal to a mathematical model, to compute the theoretical diffraction spectrum and to compare it with the experimental data.

1. Introduction

After the first discovery of an icosahedral quasicrystal [29] various models were proposed to describe its structure [4, 12, 13, 15, 27, 33]. They take into consideration either the existence of well-defined atomic clusters or the good enough agreement of the Penrose pattern with the experimental data as concerns the orientational and translational properties, Fourier spectrum, etc. Generally, a quasicrystal is regarded as either a hierarchical packing of atomic clusters [2, 19, 20] or as a decorated quasiperiodic tiling obtained by projection [15, 16].

In the case of the first class of models, well-defined atomic clusters are packed quasiperiodically into hierarchical aggregates following some inflation rules. For example, the basic element of the structure used in the model of Janot and de Boissieu [19, 20], called pseudo-Mackay icosahedra (PMI) is made of 42 atoms (12 vertices of an icosahedron plus 30 vertices of an icosidodecahedron) and an inner shell of eight or nine atoms distributed on the sites of a small dodecahedron. The model is generated recursively by starting from a PMI. At each step the pattern is inflated $\tau^3 = 2 + \sqrt{5}$ times, and each of its points are replaced by a PMI having the orientation of the starting PMI. In this way succesive generations of PMI are connected along twofold and threefold bondings. In order to fill the gaps between these clusters some 'connecting units' must be added and this leads to a complicated enough geometry. These 'interfaces' connecting the PMI are pieces of PMI arranged in shells having the same density as PMI, and they also obey the inflation rules of the PMI.

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The second class of models is based mainly on the Penrose tiling (also called Amman–Kramer–Neri tiling). The geometry and the Fourier spectrum of this tiling show that it may be a good basis for mathematical models. The arthmetical neighbours [21] of each vertex are distributed on the sites of a regular icosahedron, but the relative frequency of such regular icosahedrons with almost all the sites occupied is very small, and one cannot distinguish a relevant generating icosahedral cluster. For example, it is not sufficient to consider the vertex set of the Penrose tiling as the set of atomic positions in order to obtain a good model. One possibility to improve the agreement with experimental data is to decorate the tiles by using atom clusters with icosahedral symmetry [15, 16] or even individual atoms. Recently, Abe *et al* [1] discovered by using high-resolution transmission electron microscopy in Zn–Mg–rare-earth a quasicrystal whose atomic structure is very simple and can be described by decorating the Penrose tiling by individual atoms. The Penrose tiling whose tile edges are along the fivefold axes of the icosahedral symmetry is not the only tiling used in icosahedral quasicrystal modelling. A good approximation of the AlMnSi quasicrystal structure was obtained by Cheng *et al* [5] by using a decorated tiling where the tile edges are positioned along the threefold axes.

The purpose of this paper is to present a method which can be used to study the self-similarities of a model set (section 2), and a method to define model sets starting from G clusters (section 3). It is an improved variant of the method used to construct quasiperiodic patterns obtained whilst in collaboration with Verger-Gaugry [6, 7], and we think that these methods may lead to a new class of models. The usual construction of the Penrose pattern starts from a regular icosahedron and the arithmetical neighbours of each vertex are distributed on the sites of a regular icosahedron. A similar construction performed by starting from a regular dodecahedron leads [8] to a pattern in which the arithmetical neighbours of each point occupy some of the vertices of a regular dodecahedron. More generally, if we start from the cluster formed by the vertices of a regular icosahedron and a regular dodecahedron we get a pattern in which the arithmetical neighbours of each point are distributed on the sites of a regular icosahedron and a regular dodecahedron. A similar result is obtained if we add a new shell formed by the vertices of an icosidodecahedron or arbitrarily modify the radii of the shells. These icosahedral polyhedrons correspond to some orbits of the icosahedral group Y, and a quasiperiodic pattern can be obtained by starting from each Y cluster. We think that the huge number of patterns which can be defined in this way may open the possibility of obtaining some models directly, without decoration.

Our method works in the case of any finite group G, and allows us to obtain a large variety of patterns simultaneously satisfying the following three conditions:

(i) they are quasiperiodic,
(ii) they can be regarded as a packing of clusters,
(iii) they have the desired local structure.

In the case of the pattern obtained by starting from a G cluster $\mathcal{C}$, the arithmetical neighbours of each point x are distributed on sites belonging to the translation $x + \mathcal{C}$ of $\mathcal{C}$, and hence the model is a union of partially occupied translations of the G cluster $\mathcal{C}$.

A similar attempt to obtain new models was proposed by Soma and Watanabe [30, 31]. In order to use their approach (presented, up to now, only in a few concrete cases) it is necessary to first determine orthonormal bases in both the physical and internal space. Our method is simpler since it contains explicit mathematical expressions applicable to any finite group G and to any G cluster. We use only an orthonormal basis for the physical space and this can be obtained in a canonical way (theorem 6). In addition, in a natural way our formalism yields a method which can be used to study the self-similarities of the obtained pattern and of its diffraction spectrum.

2. Model sets

Let V be a finite-dimensional vector space, $\mathcal{D} \subset V$ be a lattice (that is a $\mathbb{Z}$-module generated by a basis of V), and $V_1, V_2 \subset V$ be two subspaces such that $V = V_1 \oplus V_2$. The collection of spaces and mappings

$$\begin{array}{ccccc} x \leftarrow (x, y) \quad : V_1 & \xleftarrow{\pi_1} & V_1 \oplus V_2 & \xrightarrow{\pi_2} & V_2 : \quad (x, y) \rightarrow y \\ & & \cup & & \\ & & \mathcal{D} & & \end{array} \tag{1}$$

is called a *cut and project scheme* [22, 26] if the following two conditions are satisfied:

(i) π_1 restricted to $\mathcal{D}$ is injective.
(ii) $\pi_2(\mathcal{D})$ is dense in V_2.

It is usually denoted by $(V_1 \oplus V_2, \mathcal{D})$ and allows one to define the pattern

$$\Lambda(K) = \{\pi_1(x) | x \in \mathcal{D}, \pi_2(x) \in K\} \tag{2}$$

called a *model set* [22, 24–26] for any compact set $K \subset V_2$ such that

$$K = \overline{\mathrm{int}(K)} \neq \emptyset. \tag{3}$$

Using the mapping

$$\mathcal{D}_1 \longrightarrow V_2 : x \mapsto x^{\diamond} = \pi_2((\pi_1|_{\mathcal{D}})^{-1}(x)) \tag{4}$$

where $\mathcal{D}_1 = \pi_1(\mathcal{D})$, we get

$$\mathcal{D} = \{(x, x^{\diamond}) | x \in \mathcal{D}_1\} \tag{5}$$

$$\Lambda(K) = \{x \in \mathcal{D}_1 | x^{\diamond} \in K\}. \tag{6}$$

We shall identify the space V_1 with a space $\mathbb{R}^n$ by choosing a basis $\{v_1, v_2, \ldots, v_n\}$ in V_1, and regard $\Lambda(K)$ as a subset of $\mathbb{R}^n$. The projectors $\pi_1 : V \longrightarrow V_1$ and $\pi_2 : V \longrightarrow V_2$ will be identified with the corresponding mappings $\pi_1 : V \longrightarrow V$ and $\pi_2 : V \longrightarrow V$. A *self-similarity* of $\Lambda(K)$ is an affine mapping [23]

$$A : \mathbb{R}^n \longrightarrow \mathbb{R}^n : x \mapsto Ax = \lambda x + v \tag{7}$$

where $\lambda \in \mathbb{R} - \{0\}$ and $v \in \mathbb{R}^n$ are such that

$$x \in \Lambda(K) \Longrightarrow Ax \in \Lambda(K) \tag{8}$$

that is,

$$\left. \begin{array}{r} x \in \mathcal{D}_1 \\ x^{\diamond} \in K \end{array} \right\} \Longrightarrow \left\{ \begin{array}{l} Ax \in \mathcal{D}_1 \\ (Ax)^{\diamond} \in K. \end{array} \right. \tag{9}$$

The number λ is called a *scaling factor* of $\Lambda(K)$. If $\lambda \neq 1$, then the point $a = v/(1 - \lambda)$ having the property $Aa = a$ is called an *inflation centre.*

Let $\{w_1, w_2, \ldots, w_m\}$ be a basis of $\mathcal{D}$, that is, a basis of V such that

$$\mathcal{D} = \sum_{j=1}^{m} \mathbb{Z} w_j \tag{10}$$

and let $\mathcal{A}$ be the set of all the pairs $(\lambda, \lambda') \in (\mathbb{R} - \{0\}) \times [-1, 1]$ for which the entries of the matrix of $\lambda\pi_1 + \lambda'\pi_2$ in the basis $\{w_1, w_2, \ldots, w_m\}$ are integers. We shall prove that the set

$$\mathcal{I} = \{\lambda | \text{ there exists } \lambda' \in [-1, 1] \text{ such that } (\lambda, \lambda') \in \mathcal{A}\}$$

contains some of the scaling factors of $\Lambda(K)$.

Theorem 1. *If* $(\lambda, \lambda') \in \mathcal{A}$ *and* $v \in \mathcal{D}_1$ *are such that*

$$\lambda' K + v^\diamond \subset K \tag{11}$$

then

$$A : \mathbb{R}^n \longrightarrow \mathbb{R}^n : x \mapsto Ax = \lambda x + v \tag{12}$$

is a self-similarity of the model set $\Lambda(K)$.

Proof. Since the entries of $M = \lambda\pi_1 + \lambda'\pi_2$ in the basis $\{w_1, w_2, \ldots, w_m\}$ of $\mathcal{D}$ are integers we have $M\mathcal{D} \subset \mathcal{D}$, whence

$$(x, x^\diamond) \in \mathcal{D} \Longrightarrow (\lambda x + v, \lambda' x^\diamond + v^\diamond) = M(x, x^\diamond) + (v, v^\diamond) \in \mathcal{D}.$$

Taking into acount the relation $\lambda' K + v^\diamond \subset K$ we get

$$\left.\begin{array}{r} x \in \mathcal{D}_1 \\ x^\diamond \in K \end{array}\right\} \Longrightarrow \left\{\begin{array}{c} \lambda x + v \in \mathcal{D}_1 \\ (\lambda x + v)^\diamond = \lambda' x^\diamond + v^\diamond \in K. \end{array}\right.$$

that is,

$$x \in \Lambda(K) \Longrightarrow \lambda x + v \in \Lambda(K).$$

□

We say that a set $K \subset V_2$ is a *balanced set* relative to $y \in V_2$ if

$$\{y + \alpha(x - y) | \alpha \in [-1, 1]\} \subset K \tag{13}$$

for any $x \in K$. Particularly, the projection on V_2 of a hypercube from V is a balanced set relative to the projection of its centre.

Theorem 2. *If* $K \subset V_2$ *is a balanced set relative to a point* $y^\diamond \in \pi_2(\mathcal{D})$ *then any element of* $\mathcal{I}$ *is a scaling factor of* $\Lambda(K)$.

Proof. Let

$$A : V_1 \longrightarrow V_1 \qquad Ax = y + \lambda(x - y).$$

Since

$$(Ax)^\diamond = y^\diamond + \lambda'(x^\diamond - y^\diamond) \in K$$

for any $x \in \Lambda(K)$, it follows that $Ax = y + \lambda(x - y) = \lambda x + (1 - \lambda)y$ is a self-similarity of $\Lambda(K)$. □

In the case of certain model sets, the points of the diffraction spectrum with a brightness above a given threshold also form a model set. Thus, the method presented may allow us to determine certain self-similarities of the 'bright' part of the diffraction spectrum of model sets. Let $(V_1 \oplus V_2, \mathcal{D})$ be a cut and project scheme, $\mathcal{D}^*$ be the dual lattice of $\mathcal{D}$, and let $\mathcal{D}_1^* = \pi_1(\mathcal{D}^*)$. Assuming that $(V_1 \oplus V_2, \mathcal{D}^*)$ is also a cut and project scheme, we shall consider the mapping

$$\mathcal{D}_1^* \longrightarrow V_2 : x \mapsto x^\triangleleft = \pi_2((\pi_1|_{\mathcal{D}^*})^{-1}(x)). \tag{14}$$

If K is Riemann integrable, then [17, 18] the measure

$$\nu_K = \sum_{x \in \Lambda(K)} \delta_x \tag{15}$$

has a unique autocorrelation γ_K, and the atomic part of the Fourier transform of γ_K is

$$\hat{\gamma}_K^{at} = \sum_{\xi \in \mathcal{D}_1^*} |c_\xi|^2 \delta_\xi \tag{16}$$

where $c_\xi = \hat{1}_K(-\xi^{\triangleleft})$, and 1_K is the characteristic function of K. More than that, for a suitable cut-off α,

$$\hat{\gamma}^{at}_{K,\alpha} = \sum_{\xi\in\mathcal{D}_1^*,\ |c_\xi|\geqslant\alpha} |c_\xi|^2\delta_\xi \tag{17}$$

gives [11,17] a good description of the diffraction spectrum in the case of certain quasicrystals if the terms are interpreted as describing spots with an intensity proportional to $|c_\xi|^2$.

Let $\{w'_1, w'_2, \ldots, w'_m\}$ be a basis of the lattice $\mathcal{D}^*$, $\mathcal{A}^*$ be the set of all the pairs $(\lambda, \lambda') \in (\mathbb{R} - \{0\}) \times [-1, 1]$ for which the entries of the matrix of $\lambda\pi_1 + \lambda'\pi_2$ in the basis $\{w'_1, w'_2, \ldots, w'_m\}$ are integers, and let

$$\mathcal{I}^* = \{\lambda | \text{ there exists } \lambda' \in [-1, 1] \text{ such that } (\lambda, \lambda') \in \mathcal{A}^*\}.$$

Choosing $\alpha \in (0, \infty)$ such that

$$\Omega_\alpha = \{\xi \in V_2 | |\hat{1}_K(-\xi)| \geqslant \alpha\} \neq \emptyset \tag{18}$$

from theorem 1 we get the following result.

Theorem 3. *If* $(\lambda, \lambda') \in \mathcal{A}^*$ *and* $v \in \mathcal{D}_1^*$ *are such that*

$$\lambda'\Omega_\alpha + v^{\triangleleft} \subset \Omega_\alpha \tag{19}$$

then

$$A : \mathbb{R}^n \longrightarrow \mathbb{R}^n : x \mapsto Ax = \lambda x + v \tag{20}$$

is a self-similarity of the support

$$\Lambda(K)^*_\alpha = \{\xi \in \mathcal{D}_1^* | |c_\xi| \geqslant \alpha\} \tag{21}$$

of $\hat{\gamma}^{at}_{K,\alpha}$, *that is,*

$$\xi \in \Lambda(K)^*_\alpha \Longrightarrow A\xi = \lambda\xi + v \in \Lambda(K)^*_\alpha. \tag{22}$$

Example. Let $\tau = (1 + \sqrt{5})/2$, $\tau' = (1 - \sqrt{5})/2$, and let us consider the cut and project scheme

$$\begin{array}{ccccccc} x_1 \leftarrow (x_1, x_2) & : V_1 & \xleftarrow{\pi_1} & \mathbb{R}^2 & \xrightarrow{\pi_2} & V_2 : & (x_1, x_2) \to x_2 \\ & & & \cup & & & \\ & & & \mathcal{D} & & & \end{array} \tag{23}$$

where $V_1 = \mathbb{R} \equiv \{(x_1, 0) | x_1 \in \mathbb{R}\}$, $V_2 = \mathbb{R} \equiv \{(0, x_2) | x_2 \in \mathbb{R}\}$, and

$$\mathcal{D} = \{(j + m\tau,\ j + m\tau') | j, m \in \mathbb{Z}\} = \mathbb{Z}(1, 1) + \mathbb{Z}(\tau, \tau').$$

In this case $\mathcal{D}_1 = \{(j + m\tau, 0) \mid j, m \in \mathbb{Z}\} = \mathbb{Z} + \mathbb{Z}\tau$, and $(j + m\tau)^\diamond = j + m\tau'$. The vectors $w_1 = (1, 1)$ and $w_2 = (\tau, \tau')$ form a basis of $\mathcal{D}$, and the matrices of π_1 and π_2 in the basis $\{w_1, w_2\}$ are

$$\pi_1 = \mathcal{M}((5 - \sqrt{5})/10, (5 + \sqrt{5})/10, \sqrt{5}/5)$$
$$\pi_2 = \mathcal{M}((5 + \sqrt{5})/10, (5 - \sqrt{5})/10, -\sqrt{5}/5)$$

where

$$\mathcal{M}(\alpha, \beta, \gamma) = \begin{pmatrix} \alpha & \gamma \\ \gamma & \beta \end{pmatrix}.$$

Since the matrix of $\lambda\pi_1 + \lambda'\pi_2$ in the basis $\{w_1, w_2\}$ has integer entries if and only if

$$j = \lambda\frac{5-\sqrt{5}}{10} + \lambda'\frac{5+\sqrt{5}}{10} \in \mathbb{Z}$$
$$m = \lambda\frac{\sqrt{5}}{5} - \lambda'\frac{\sqrt{5}}{5} \in \mathbb{Z}$$
$$l = \lambda\frac{5+\sqrt{5}}{10} + \lambda'\frac{5-\sqrt{5}}{10} \in \mathbb{Z}$$

we get

$$\mathcal{I} = \{j + m\tau \,|\, j, m \in \mathbb{Z},\ j + m\tau' \in [-1, 1]\}.$$

Let $r \in (0, \infty)$, and let

$$\Lambda([-r, r]) = \{j + m\tau \in \mathcal{D}_1 \,|\, j + m\tau' \in [-r, r]\}$$

be the model set corresponding to the window $K = [-r, r]$. For any $\lambda \in \mathcal{I}$ and any $v \in \mathcal{D}_1$ such that $|v^\diamond| \leqslant (1 - |\lambda^\diamond|)r$ the transformation

$$\mathbb{R} \longrightarrow \mathbb{R} : x \mapsto \lambda x + v$$

is a self-similarity of $\Lambda([-r, r])$.

The dual lattice of $\mathcal{D}$ is $\mathcal{D}^* = \mathbb{Z}w'_1 + \mathbb{Z}w'_2$, where

$$w'_1 = \left(\frac{1}{\tau+2}, \frac{\tau+1}{\tau+2}\right) \qquad w'_2 = \left(\frac{\tau}{\tau+2}, \frac{-\tau}{\tau+2}\right)$$

and $(V_1 \oplus V_2, \mathcal{D}^*)$ is also a cut and project scheme. Since the matrices of π_1 and π_2 in the basis $\{w'_1, w'_2\}$ coincide with the corresponding matrices in the basis $\{w_1, w_2\}$, it follows that $\mathcal{I}^* = \mathcal{I}$.

In this case, the Fourier transform $\hat{1}_{[-r,r]}$ of $1_{[-r,r]}$ is

$$\hat{1}_{[-r,r]}(\xi) = \frac{\sin(2\pi r\xi)}{\pi\xi}$$

and the maximal value of $|\hat{1}_{[-r,r]}(\xi)|$ is $2r = \lim_{\xi\to 0} |\hat{1}_{[-r,r]}(\xi)|$. For any β in a certain neighbourhood of 0 and $\alpha = \hat{1}_{[-r,r]}(\beta)$ we get $\Omega_\alpha = [-\beta, \beta]$. In this case, any $\lambda \in \mathcal{I}^*$ is a scaling factor of the diffraction spectrum of $\Lambda([-r, r])$ if we take into consideration only the spots $\xi \in \mathcal{D}_1^*$ with $|c_\xi| \geqslant \alpha$.

We think that similar results can be obtained for the three-dimensional Penrose tiling [9], and in the case of many other known patterns.

3. *G*-model sets

In this section we present the construction of G-model sets. Let

$$\{g : \mathbb{E}_n \longrightarrow \mathbb{E}_n \,|\, g \in G\}$$

be an orthogonal $\mathbb{R}$-irreducible faithful representation of a finite group G in the usual n-dimensional Euclidean space $\mathbb{E}_n = (\mathbb{R}^n, \langle,\rangle)$, and let $S \subset \mathbb{E}_n$ be a finite non-empty set which does not contain the null vector. Any finite union of orbits of G is called a *G cluster.* Particularly,

$$\mathcal{C} = \bigcup_{r\in S} Gr \cup \bigcup_{r\in S} G(-r) = \{e_1, e_2, \ldots, e_k, -e_1, -e_2, \ldots, -e_k\} \qquad (24)$$

is the G-cluster symmetric with respect to the origin generated by S. For each $g \in G$, there exist the numbers $s_1^g,\ s_2^g, \ldots, s_k^g \in \{-1;\ 1\}$ and a permutation of the set $\{1, 2, \ldots, k\}$ also denoted by g such that

$$ge_j = s^g_{g(j)} e_{g(j)} \tag{25}$$

for any $j \in \{1, 2, \ldots, k\}$.

If we start from the representation

$$\begin{aligned} a(x, y, z) &= \left(\frac{1}{2}x - \frac{\tau}{2}y + \frac{\tau-1}{2}z, \frac{\tau}{2}x + \frac{\tau-1}{2}y - \frac{1}{2}z, \frac{\tau-1}{2}x + \frac{1}{2}y + \frac{\tau}{2}z\right) \\ b(x, y, z) &= (-x, -y, z) \end{aligned} \tag{26}$$

of the icosahedral group $Y = 235 = \langle a, b | a^5 = b^2 = (ab)^3 = e\rangle$ and the set $S = \{(1, 0, \tau)\}$ then $\mathcal{C} = \{e_1, e_2, \ldots, e_6, -e_1, -e_2, \ldots, -e_6\}$, where

$$\begin{array}{lll} e_1 = (1, 0, \tau) & e_3 = (\tau, 1, 0) & e_5 = (-1, 0, \tau) \\ e_2 = (\tau, -1, 0) & e_4 = (0, \tau, 1) & e_6 = (0, -\tau, 1). \end{array} \tag{27}$$

For $g = a$ and $g = b$ relation (25) can be written as

$$a = \begin{pmatrix} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\ e_1 & e_3 & e_4 & e_5 & e_6 & e_2 \end{pmatrix} \qquad b = \begin{pmatrix} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\ e_5 & -e_2 & -e_3 & e_6 & e_1 & e_4 \end{pmatrix} \tag{28}$$

and the corresponding Y-model set is the usual three-dimensional Penrose pattern [9]. Many other concrete illustrations of our results can be found in [7, 8].

Theorem 4 ([6–8]). *The group G can be identified with the group of permutations*

$$\{\mathcal{C} \longrightarrow \mathcal{C} : r \mapsto gr | g \in G\}$$

and the formula

$$g(x_1, \ldots, x_k) = (s_1^g x_{g^{-1}(1)}, s_2^g x_{g^{-1}(2)}, \ldots, s_k^g x_{g^{-1}(k)}) \tag{29}$$

defines an orthogonal representation of G in $\mathbb{E}_k$.

Theorem 5 ([6–8]). *The subspaces*

$$\mathbb{E}_k^{\parallel} = \{(\langle r, e_1\rangle, \langle r, e_2\rangle, \ldots, \langle r, e_k\rangle) | r \in \mathbb{E}_n\} \tag{30}$$

$$\mathbb{E}_k^{\perp} = \left\{(x_1, x_2, \ldots, x_k) \in \mathbb{E}_k \;\middle|\; \sum_{i=1}^{k} x_i e_i = 0\right\} \tag{31}$$

of $\mathbb{E}_k$ are G-invariant, orthogonal, and $\mathbb{E}_k = \mathbb{E}_k^{\parallel} \oplus \mathbb{E}_k^{\perp}$.

Let $u_1 = (1, 0, 0, \ldots, 0), u_2 = (0, 1, 0, \ldots, 0), \ldots, u_n = (0, \ldots, 0, 1)$ be the canonical basis of $\mathbb{E}_n$, and let

$$e_j = (e_{j1}, e_{j2}, \ldots, e_{jn}) \tag{32}$$

for any $j \in \{1, 2, \ldots, k\}$.

Theorem 6 ([6–8]). *The vectors $v_1', v_2', \ldots, v_n'$, where*

$$v_i' = (e_{1i}, e_{2i}, \ldots, e_{ki}) \tag{33}$$

form an orthogonal basis of $\mathbb{E}_k^{\parallel}$, and

$$\|v_1'\| = \|v_2'\| = \cdots = \|v_n'\|. \tag{34}$$

The orthonormal basis of $\mathbb{E}_k^{\parallel}$ corresponding to $\{v_1', v_2', \ldots, v_n'\}$ is formed by the vectors

$$v_1 = \varrho v_1' \qquad v_2 = \varrho v_2' \ldots v_n = \varrho v_n' \tag{35}$$

where $\varrho = 1/\|v_1'\|$.

Theorem 7 ([6–8]). *The representation of G in $\mathbb{E}_k^{\parallel}$ is equivalent to the representation of G in $\mathbb{E}_n$, and the isomorphism*

$$\Xi : \mathbb{E}_n \longrightarrow \mathbb{E}_k^{\parallel} \qquad \Xi r = (\varrho\langle r, e_1\rangle, \varrho\langle r, e_2\rangle, \ldots, \varrho\langle r, e_k\rangle) \tag{36}$$

having the property $\Xi u_i = v_i$ allows us to identify the two spaces.

Theorem 8 ([6–8]). *The mapping $\pi^{\parallel} : \mathbb{E}_k \longrightarrow \mathbb{E}_k$*

$$\pi^{\parallel}(x_1, \ldots, x_k) = \left(\varrho^2 \sum_{i=1}^{k} \langle e_1, e_i\rangle x_i, \ldots, \varrho^2 \sum_{i=1}^{k} \langle e_k, e_i\rangle x_i\right) \tag{37}$$

is the orthogonal projector corresponding to the subspace $\mathbb{E}_k^{\parallel}$.

Theorem 9 ([6–8]). *The $\mathbb{Z}$-module*

$$\mathcal{L} = \kappa\mathbb{Z}^k \subset \mathbb{E}_k \tag{38}$$

where $\kappa = 1/\varrho$, is G-invariant, and in view of the identification $\Xi : \mathbb{E}_n \longrightarrow \mathbb{E}_k^{\parallel}$, we have

$$\pi^{\parallel}\mathcal{L} = \sum_{i=1}^{k} \mathbb{Z}e_i. \tag{39}$$

Generally,

$$\begin{array}{ccccc} \mathbb{E}_k^{\parallel} & \xleftarrow{\pi^{\parallel}} & \mathbb{E}_k^{\parallel} \oplus \mathbb{E}_k^{\perp} & \xrightarrow{\pi^{\perp}} & \mathbb{E}_k^{\perp} \\ & & \cup & & \\ & & \mathcal{L} & & \end{array} \tag{40}$$

where $\pi^{\perp} = 1 - \pi^{\parallel}$, is not a cut and project scheme since, generally, $\pi^{\parallel}$ restricted to $\mathcal{L}$ is not injective.

Theorem 10 ([10, 28]). *Any $\mathbb{Z}$-module $L \subset \mathbb{R}^l$ is the direct sum of a lattice L_d of rank d and a $\mathbb{Z}$-module L_s dense in a vector subspace of dimension s, where $d + s$ is the dimension of the subspace generated by L in $\mathbb{R}^l$.*

The $\mathbb{Z}$-module $\mathcal{L}^{\perp} = \pi^{\perp}(\mathcal{L})$ is the direct sum $\mathcal{L}^{\perp} = \mathcal{L}_s^{\perp} \oplus \mathcal{L}_d^{\perp}$ of a lattice $\mathcal{L}_d^{\perp}$ of rank d and a $\mathbb{Z}$-module $\mathcal{L}_s^{\perp}$ dense in a subspace $V_2 \subset \mathbb{E}_k^{\perp}$ of dimension s, where $d + s = \dim \mathbb{E}_k^{\perp}$. In this decomposition the space V_2 is uniquely determined.

Theorem 11. *The space V_2 is a G-invariant subspace of $\mathbb{E}_k^{\perp}$.*

Proof. We have to prove that $y \in V_2 \implies g(y) \in V_2$, for any $g \in G$. Since $\pi^{\perp}(g(x)) = g(\pi^{\perp}x)$ for any $x \in \mathbb{E}_k$, it follows that $\mathcal{L}^{\perp}$ is a G-invariant $\mathbb{Z}$-module. If $y \in V_2$ then there exists a sequence $(x_j)_{j\geqslant 0} \subset \mathcal{L}^{\perp}$ such that $y = \lim_{j\to\infty} x_j$. Since g is an isometry and $g(x_j) \in \mathcal{L}^{\perp}$ for any $j \in \mathbb{N}$, it follows that $g(y) = \lim_{j\to\infty} g(x_j) \in V_2$. $\square$

This theorem shows that a decomposition of the representation of G in $\mathbb{E}_k^{\perp}$ into $\mathbb{R}$-irreducible representations may help us to determine the subspace V_2.

Let $V_1 = \mathbb{E}_k^{\parallel}$, and let

$$W = \{x \in \mathbb{E}_k^{\perp} | \langle x, y\rangle = 0 \text{ for any } y \in V_2\} \tag{41}$$

be the orthogonal of V_2 in $\mathbb{E}_k^\perp$. For each $x \in \mathbb{E}_k$ there exist $x^\parallel \in V_1$, $x' \in V_2$ and $x'' \in W$ uniquely determined such that $x = x^\parallel + x' + x''$. The mappings

$$\pi' : \mathbb{E}_k \longrightarrow \mathbb{E}_k : \ x \mapsto x' \qquad \pi'' : \mathbb{E}_k \longrightarrow \mathbb{E}_k : \ x \mapsto x''$$

are the orthogonal projectors corresponding to the subspaces V_2 and W. Let $\varepsilon_1 = (1, 0, \ldots, 0)$, $\varepsilon_2 = (0, 1, 0, \ldots, 0), \ldots, \varepsilon_k = (0, 0, \ldots, 0, 1)$ be the canonical basis of $\mathbb{E}_k$.

Theorem 12. *The matrix of π'' in the basis $\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k\}$ has rational entries.*

Proof. Let

$$\pi'' = \begin{pmatrix} d_{11} & d_{12} & \ldots & d_{1k} \\ d_{21} & d_{22} & \ldots & d_{2k} \\ \ldots & \ldots & \ldots & \ldots \\ d_{k1} & d_{k2} & \ldots & d_{kk} \end{pmatrix}$$

be the matrix of π'' in the basis $\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k\}$, that is, $\pi''\varepsilon_j = \sum_{i=1}^k d_{ij}\varepsilon_i$. The matrix π'' is symmetric

$$d_{ij} = \langle \pi''\varepsilon_j, \varepsilon_i \rangle = \langle \pi''\varepsilon_j, \pi''\varepsilon_i \rangle = \langle \varepsilon_j, \pi''\varepsilon_i \rangle = d_{ji}.$$

We start by proving that the ratio of any two non-zero entries lying on the same column of π'' is a rational number. In order to simplify the notations we prove that our statement is valid for the first column, but our arguments work for any column.

Let us suppose that the first column contains two $\mathbb{Q}$-linearly independent entries. If this number is larger then the proof is similar. In this case, there exist two $\mathbb{Q}$-linearly independent numbers ω and ν and the integers $\gamma, \alpha_1, \beta_1, \ldots, \alpha_k, \beta_k$ such that

$$d_{j1} = \frac{\alpha_j}{\gamma}\omega + \frac{\beta_j}{\gamma}\nu$$

for any $j \in \{1, 2, \ldots, k\}$.

From the relation $\pi'' \circ \pi'' = \pi''$ we get

$$d_{11} = (d_{11})^2 + (d_{21})^2 + \cdots + (d_{k1})^2$$

whence $d_{11} \neq 0$. Thus, we can assume that $\omega = \gamma d_{11}$. It follows:

$$\pi''\varepsilon_1 = \frac{\omega}{\gamma}(\varepsilon_1 + \alpha_2\varepsilon_2 + \cdots + \alpha_k\varepsilon_k) + \frac{\nu}{\gamma}(\beta_2\varepsilon_2 + \cdots + \beta_k\varepsilon_k). \tag{42}$$

The equality $\pi'' \circ \pi'' = \pi''$ allows us to write this relation in the form

$$\pi''\varepsilon_1 = \frac{\omega}{\gamma}\pi''(\varepsilon_1 + \alpha_2\varepsilon_2 + \cdots + \alpha_k\varepsilon_k) + \frac{\nu}{\gamma}\pi''(\beta_2\varepsilon_2 + \cdots + \beta_k\varepsilon_k)$$

or

$$(\omega - \gamma)\pi''\varepsilon_1 + \omega\pi''(\alpha_2\varepsilon_2 + \cdots + \alpha_k\varepsilon_k) + \nu\pi''(\beta_2\varepsilon_2 + \cdots + \beta_k\varepsilon_k) = 0.$$

Let us first consider the case when the numbers $\omega - \gamma$, ω and ν are $\mathbb{Q}$-linearly independent. In view of Kronecker's theorem ([24], p 286), for any $\delta > 0$ there exist a real number t and the integers η_1, η_2, η_3 such that

$$|t(\omega - \gamma) - \eta_1| \leqslant \delta \qquad |t\omega - \eta_2| \leqslant \delta \qquad |t\nu - \eta_3| \leqslant \delta.$$

This means that there exists the vector

$$y = \kappa[\eta_1\varepsilon_1 + \eta_2(\alpha_2\varepsilon_2 + \cdots + \alpha_k\varepsilon_k) + \eta_3(\beta_2\varepsilon_2 + \cdots + \beta_k\varepsilon_k)] \in \mathcal{L}$$

such that

$$\begin{aligned}\|\pi''y\| = \kappa \|(\eta_1 - t(\omega-\gamma))\pi''\varepsilon_1 + (\eta_2 - t\omega)\pi''(\alpha_2\varepsilon_2 + \cdots + \alpha_k\varepsilon_k)\\ +(\eta_3 - t\nu)\pi''(\beta_2\varepsilon_2 + \cdots + \beta_k\varepsilon_k)\|\\ \leqslant \kappa\delta(\|\pi''\varepsilon_1\| + \|\pi''(\alpha_2\varepsilon_2 + \cdots + \alpha_k\varepsilon_k)\| + \|\pi''(\beta_2\varepsilon_2 + \cdots + \beta_k\varepsilon_k)\|).\end{aligned}$$

Since $\pi''(\mathcal{L}) \subset W$ is a discrete set, we must have $\pi''y = 0$ for any δ small enough. This is possible only if

$$\begin{aligned}&\pi''\varepsilon_1 = 0\\ &\pi''(\alpha_2\varepsilon_2 + \cdots + \alpha_k\varepsilon_k) = 0\\ &\pi''(\beta_2\varepsilon_2 + \cdots + \beta_k\varepsilon_k) = 0.\end{aligned} \tag{43}$$

Since (43) contradicts (42), the numbers $\omega - \gamma$, ω, ν cannot be $\mathbb{Q}$-linearly independent. There exist the integers ζ, ξ and θ such that

$$\omega - \gamma = \frac{\zeta}{\theta}\omega + \frac{\xi}{\theta}\nu$$

and hence,

$$\omega\pi''(\zeta\varepsilon_1 + \theta\alpha_2\varepsilon_2 + \cdots + \theta\alpha_k\varepsilon_k) + \nu\pi''(\xi\varepsilon_1 + \theta\beta_2\varepsilon_2 + \cdots + \theta\beta_k\varepsilon_k) = 0.$$

In view of Kronecker's theorem, for any $\delta > 0$, there exist a real number t and the integers μ_1, μ_2, such that

$$|t\omega - \mu_1| \leqslant \delta \qquad |t\nu - \mu_2| \leqslant \delta.$$

These relations show that there exists

$$y = \kappa[\mu_1(\zeta\varepsilon_1 + \theta\alpha_2\varepsilon_2 + \cdots + \theta\alpha_k\varepsilon_k) + \mu_2(\xi\varepsilon_1 + \theta\beta_2\varepsilon_2 + \cdots + \theta\beta_k\varepsilon_k)] \in \mathcal{L}$$

such that

$$\|\pi''y\| \leqslant \kappa\delta(\|\pi''(\zeta\varepsilon_1 + \theta\alpha_2\varepsilon_2 + \cdots + \theta\alpha_k\varepsilon_k)\| + \|\pi''(\xi\varepsilon_1 + \theta\beta_2\varepsilon_2 + \cdots + \theta\beta_k\varepsilon_k)\|).$$

Since $\pi''(\mathcal{L}) \subset W$ is a discrete set, we must have $\pi''y = 0$ for any δ small enough. This is only possible if

$$\begin{aligned}&\pi''(\zeta\varepsilon_1 + \theta\alpha_2\varepsilon_2 + \cdots + \theta\alpha_k\varepsilon_k) = 0\\ &\pi''(\xi\varepsilon_1 + \theta\beta_2\varepsilon_2 + \cdots + \theta\beta_k\varepsilon_k) = 0.\end{aligned}$$

If we multiply scalarly the second relation by ε_1 we get

$$\xi\omega + \theta\beta_2(\omega\alpha_2 + \nu\beta_2) + \cdots + \theta\beta_k(\omega\alpha_k + \nu\beta_k) = 0$$

that is,

$$\omega(\xi + \theta\alpha_2\beta_2 + \cdots + \theta\alpha_k\beta_k) + \theta\nu[(\beta_2)^2 + \cdots + (\beta_k)^2] = 0.$$

Since ω and ν are $\mathbb{Q}$-linearly independent, we obtain

$$(\beta_2)^2 + \cdots + (\beta_k)^2 = 0$$

that is,

$$\beta_2 = \beta_3 = \cdots = \beta_k = 0.$$

It follows that $d_{j1} = \alpha_j\omega$ for any $j \in \{1, 2, \ldots, k\}$, and this result contradicts the assumption that the first column contains two $\mathbb{Q}$-linearly independent entries. Hence, the ratio of any two non-zero entries lying on the same column is a rational number. Since our matrix is symmetric, the ratio of any two non-zero entries lying on the same row is also a rational number. This is possible only if π'' is the product of a matrix with rational entries M and a real number ω, that is, $\pi'' = \omega M$. The equality $\pi'' \circ \pi'' = \pi''$ shows that there exists a rational number α such that $\omega^2 = \alpha\omega$, and hence ω is also a rational number. □

Let

$$V = V_1 \oplus V_2 \qquad \mathcal{D} = p(\mathcal{L}) \tag{44}$$

where

$$p : \mathbb{E}_k \longrightarrow \mathbb{E}_k : x \mapsto x^{\parallel} + x' \tag{45}$$

and let

$$\pi_1 : V \longrightarrow V : x \mapsto x^{\parallel} \qquad \pi_2 : V \longrightarrow V : x \mapsto x'. \tag{46}$$

Theorem 13. *For any G cluster $\mathcal{C}$ the collection of spaces and mappings*

$$\begin{array}{ccccc} V_1 & \xleftarrow{\pi_1} & V_1 \oplus V_2 & \xrightarrow{\pi_2} & V_2 \\ & & \cup & & \\ & & \mathcal{D} & & \end{array} \tag{47}$$

is a cut and project scheme. In addition,

$$\pi_1(\mathcal{D}) = \sum_{j=1}^{k} \mathbb{Z} e_j. \tag{48}$$

Proof. We have to prove that $\mathcal{D}$ is a lattice in V, the restriction of π_1 to $\mathcal{D}$ is injective, and $\pi_2(\mathcal{D})$ is dense in V_2.

Since $p = 1 - \pi''$ has rational entries and its rank coincides with the dimension of V, it follows that $\mathcal{D} = p(\mathcal{L})$ is a lattice in V.

From the relation $\pi_2(\mathcal{D}) = \pi'(p(\mathcal{L})) = \pi'(\mathcal{L})$ and the definition of V_2, it follows that $\pi_2(\mathcal{D})$ is dense in V_2.

It remains to prove that π_1 restricted to $\mathcal{D}$ is injective. For this it is sufficient to prove that $\mathrm{Ker}(\pi_1|_{\mathcal{D}}) = \{0\}$ and this is equivalent to proving that $\mathrm{Ker}(\pi^{\parallel}|_{\mathcal{L}}) \subset W$. We have to analyse the case $\mathrm{Ker}(\pi^{\parallel}|_{\mathcal{L}}) \neq \{0\}$. Let $x \in \mathcal{L} - \{0\}$ be such that $\pi^{\parallel} x = 0$. The relation $\mathrm{Ker}\,\pi^{\parallel} = \mathbb{E}_k^{\perp}$ shows that $x \in \mathcal{L} \cap \mathbb{E}_k^{\perp}$. For any $y \in \mathcal{L}$ the points $(z_1, z_2, \ldots, z_k) \in \mathbb{E}_k$ satisfying the equation

$$x_1(z_1 - y_1) + x_2(z_2 - y_2) + \cdots + x_k(z_k - y_k) = 0 \tag{49}$$

form the hyperplane H_y orthogonal to x passing through y. The hyperplane H_y intersects the one-dimensional subspace $\mathbb{R}x = \{\alpha x | \alpha \in \mathbb{R}\}$ at a point corresponding to $\alpha = (x_1 y_1 + \cdots + x_k y_k)/(x_1^2 + \cdots + x_k^2) = \langle x, y\rangle / \|x\|^2$.

Since $\langle x, y\rangle \in \kappa^2\mathbb{Z}$, the minimal distance between two distinct hyperplanes of the family of parallel hyperplanes $\{H_y \mid y \in \mathcal{L}\}$ is $\kappa^2/\|x\|$. The union $H = \bigcup_{y\in\mathcal{L}} H_y$ of hyperplanes orthogonal to $x \in \mathbb{E}_k^{\perp}$ contains the $\mathbb{Z}$-module $\mathcal{L}^{\perp} = \pi^{\perp}(\mathcal{L})$. We prove that $x \in W$, since in the contrary case, $V_2 \cap \mathcal{L}^{\perp}$ cannot be dense in V_2. Indeed, assuming that $x \notin W$ there exists $z \in V_2$ such that $\langle x, z\rangle \neq 0$. In this case the one-dimensional subspace $\mathbb{R}z = \{\alpha z | \alpha \in \mathbb{R}\}$ of V_2 is not contained in H. The intersection $H \cap \mathbb{R}z$ is a discrete set, and for any $u \in \mathbb{R}z - H$ the distance δ between u and H is strictly positive, and hence the open ball of centre u and radius δ does not contain any element of $\mathcal{L}^{\perp}$. It remains that $x \in W$. □

A model set obtained by using a cut and project scheme $(V_1 \oplus V_2, \mathcal{D})$ defined by a G cluster is called a *G-model set*.

Example. The relations

$$a(x, y) = (cx - sy, sx + cy) \qquad b(x, y) = (x, -y) \tag{50}$$

where $c = \cos(\pi/5) = (1+\sqrt{5})/4$, $s = \sin(\pi/5) = \sqrt{5-\sqrt{5}}/(2\sqrt{2})$ define the usual two-dimensional representation of the dihedral group

$$D_{10} = \langle a, b | a^{10} = b^2 = (ab)^2 = e\rangle.$$

Let $c' = \cos(2\pi/5) = (\sqrt{5}-1)/4$, $s' = \sin(2\pi/5) = \sqrt{5+\sqrt{5}}/(2\sqrt{2})$. The D_{10}-cluster generated by the set $S = \{(1,0)\}$

$$\mathcal{C} = D_{10}(1,0) = \{e_1, e_2, e_3, e_4, e_5, -e_1, -e_2, -e_3, -e_4, -e_5\}$$

where $e_1 = (1,0)$, $e_2 = (c', s')$, $e_3 = (-c, s)$, $e_4 = (-c, -s)$, $e_5 = (c', -s')$, defines the orthogonal representation

$$\begin{aligned} &a(x_1, x_2, x_3, x_4, x_5) = (-x_3, -x_4, -x_5, -x_1, -x_2)\\ &b(x_1, x_2, x_3, x_4, x_5) = (x_1, x_5, x_4, x_3, x_2) \end{aligned} \tag{51}$$

of D_{10} in $\mathbb{E}_5$.

The vectors $v_1 = \varrho(1, c', -c, -c, c')$, $v_2 = \varrho(0, s', s, -s, -s')$, where $\varrho = \sqrt{2/5}$, form an orthonormal basis of the D_{10}-invariant space

$$\mathbb{E}_5^{\parallel} = \{(\langle r, e_1\rangle, \langle r, e_2\rangle, \ldots, \langle r, e_5\rangle) | r \in \mathbb{E}_2\} \tag{52}$$

and the isometry (which is an isomorphism of representations)

$$\Xi : \mathbb{E}_2 \longrightarrow \mathbb{E}_5^{\parallel} : r \mapsto (\varrho\langle r, e_1\rangle, \varrho\langle r, e_2\rangle, \ldots, \varrho\langle r, e_5\rangle) \tag{53}$$

with the property $\Xi(1,0) = v_1$, $\Xi(0,1) = v_2$ allows us to identify the two spaces.

In this case the orthogonal $\mathbb{E}_5^{\perp}$ of $\mathbb{E}_5^{\parallel}$ contains a D_{10}-invariant subspace, namely, the space $\{(x_1, x_2, \ldots, x_5) \in \mathbb{E}_5 | x_1 = x_2 = \cdots = x_5\}$. The space $\mathbb{E}_5^{\perp}$ can be decomposed into an orthogonal sum of two D_{10}-invariant subspaces. We shall denote the corresponding projectors by $\pi_1^{\perp}$ and $\pi_2^{\perp}$. The matrices of these projectors in the canonical basis of $\mathbb{E}_6$ are

$$\begin{aligned} &\pi^{\parallel} = \mathcal{M}(\tfrac{2}{5}, (\sqrt{5}-1)/10, -(\sqrt{5}+1)/10)\\ &\pi_1^{\perp} = \mathcal{M}(\tfrac{2}{5}, -(\sqrt{5}+1)/10, (\sqrt{5}-1)/10)\\ &\pi_2^{\perp} = \mathcal{M}(\tfrac{1}{5}, \tfrac{1}{5}, \tfrac{1}{5}) \end{aligned} \tag{54}$$

where

$$\mathcal{M}(\alpha, \beta, \gamma) = \begin{pmatrix} \alpha & \beta & \gamma & \gamma & \beta\\ \beta & \alpha & \beta & \gamma & \gamma\\ \gamma & \beta & \alpha & \beta & \gamma\\ \gamma & \gamma & \beta & \alpha & \beta\\ \beta & \gamma & \gamma & \beta & \alpha \end{pmatrix}. \tag{55}$$

Let $\kappa = 1/\varrho$ and let $\mathcal{L} = \kappa\mathbb{Z}^5$. The pair $(\mathbb{E}_5^{\parallel} \oplus \mathbb{E}_5^{\perp}, \mathcal{L})$ is not a cut and project scheme since $\pi^{\parallel}x = 0$ for any $x = (x_1, x_2, \ldots, x_5) \in \mathcal{L}$ such that $x_1 = x_2 = \cdots = x_5$. We use theorem 13 in order to obtain a cut and project scheme. In this case

$$V_1 = \mathbb{E}_5^{\parallel} \qquad V_2 = \mathbb{E}_{5,1}^{\perp} \qquad V = V_1 \oplus V_2 \qquad W = \mathbb{E}_{5,2}^{\perp} \tag{56}$$

$$p = \pi^{\parallel} + \pi_1^{\perp} \qquad \mathcal{D} = p(\mathcal{L}) \qquad \pi_1 x = \pi^{\parallel} x \qquad \pi_2 x = \pi_1^{\perp} x \tag{57}$$

and

$$\mathcal{D} = \sum_{j=1}^{4} \mathbb{Z} w_j$$

where

$$\begin{aligned} &w_1 = (\tfrac{4}{5}, -\tfrac{1}{5}, -\tfrac{1}{5}, -\tfrac{1}{5}, -\tfrac{1}{5}) \qquad w_2 = (-\tfrac{1}{5}, \tfrac{4}{5}, -\tfrac{1}{5}, -\tfrac{1}{5}, -\tfrac{1}{5})\\ &w_3 = (-\tfrac{1}{5}, -\tfrac{1}{5}, \tfrac{4}{5}, -\tfrac{1}{5}, -\tfrac{1}{5}) \qquad w_4 = (-\tfrac{1}{5}, -\tfrac{1}{5}, -\tfrac{1}{5}, \tfrac{4}{5}, -\tfrac{1}{5}). \end{aligned} \tag{58}$$

From theorem 13 it follows that $(V_1 \oplus V_2, \mathcal{D})$ is a cut and project scheme. Since the matrices corresponding to $\pi_1 : V \longrightarrow V$, $\pi_2 : V \longrightarrow V$ in the basis $\{w_1, w_2, w_3, w_4\}$ are

$$\pi_1 = \mathcal{M}'((5-\sqrt{5})/10, (5+\sqrt{5})/10, \sqrt{5}/5)$$
$$\pi_2 = \mathcal{M}'((5+\sqrt{5})/10, (5-\sqrt{5})/10, -\sqrt{5}/5)$$

where

$$\mathcal{M}'(\alpha,\beta,\gamma) = \begin{pmatrix} \alpha & \gamma & 0 & -\gamma \\ 0 & \beta & \gamma & -\gamma \\ -\gamma & \gamma & \beta & 0 \\ -\gamma & 0 & \gamma & \alpha \end{pmatrix} \tag{59}$$

the entries of the matrix $\lambda\pi_1 + \lambda'\pi_2$ are integers if and only if

$$j = \lambda\frac{5-\sqrt{5}}{10} + \lambda'\frac{5+\sqrt{5}}{10} \in \mathbb{Z}$$
$$l = \lambda\frac{5+\sqrt{5}}{10} + \lambda'\frac{5-\sqrt{5}}{10} \in \mathbb{Z}$$
$$m = \lambda\frac{\sqrt{5}}{5} - \lambda'\frac{\sqrt{5}}{5} \in \mathbb{Z}.$$

It follows

$$m = l - j \qquad \lambda = \frac{l+j}{2} + \frac{l-j}{2}\sqrt{5} \qquad \lambda' = \frac{l+j}{2} - \frac{l-j}{2}\sqrt{5} \tag{60}$$

whence

$$\mathcal{I} = \{j + m\tau \,|\, (j,m) \in \mathbb{Z}^2,\ j + m\tau' \in [-1,1]\}. \tag{61}$$

It depends on the window we choose if a certain element of $\mathcal{I}$ is or is not a scaling factor for a D_{10}-model set defined by using the cut and project scheme $(V_1 \oplus V_2, \mathcal{D})$.

The dual lattice of $\mathcal{D}$ is $\mathcal{D}^* = \sum_{j=1}^{4} \mathbb{Z}w'_j$, where

$$w'_1 = (1,0,0,0,-1) \qquad w'_2 = (0,1,0,0,-1)$$
$$w'_3 = (0,0,1,0,-1) \qquad w'_4 = (0,0,0,1,-1).$$

Since the matrices of π_1 and π_2 in the basis $\{w'_1, w'_2, w'_3, w'_4\}$ are

$$\pi_1 = \mathcal{M}''((5-\sqrt{5})/10, (5+\sqrt{5})/10, \sqrt{5}/5)$$
$$\pi_2 = \mathcal{M}''((5+\sqrt{5})/10, (5-\sqrt{5})/10, -\sqrt{5}/5)$$

where

$$\mathcal{M}''(\alpha,\beta,\gamma) = \begin{pmatrix} \alpha & 0 & -\gamma & -\gamma \\ \gamma & \beta & \gamma & 0 \\ 0 & \gamma & \beta & \gamma \\ -\gamma & -\gamma & 0 & \alpha \end{pmatrix} \tag{62}$$

we get $\mathcal{I}^* = \mathcal{I}$. An element of $\mathcal{I}^*$ may be a scaling factor for the 'bright' part of the diffraction spectrum of a model set defined by using the cut and project scheme $(V_1 \oplus V_2, \mathcal{D})$ depending on the window and the threshold of brightness we choose.

4. Conclusions

In the case of any finite group G we can obtain, in an explicit way, an infinite number of quasiperiodic patterns by using the infinite number of the corresponding G clusters. In order to obtain a mathematical model for a real quasicrystal it is sufficient to determine the corresponding symmetry group G and to approximate the local structure by using a G cluster. One can try to find some self-similarities of the model and to construct adapted wavelets bases [3, 14] useful in the description of physical properties. The existing computer programs for the cut and project method [32] allow one to compare the obtained model with the experimental data. If the agreement is not acceptable one has the possibility to search for a more suitable G cluster describing the local structure.

A quasiperiodic pattern generated by mixing lattices derived from a dodecahedral star and an icosahedral star was recently presented by Soma and Watanabe [31]. It is obtained by projection from a 16-dimensional lattice. The family of two-shell Y-clusters $Y\{(\alpha,\alpha,\alpha),(1,0,\tau)\}$ (the dodecahedron $Y(\alpha,\alpha,\alpha)$ and the icosahedron $Y(1,0,\tau)$, where α is a real positive parameter) defined by using the representation (26) of the icosahedral group Y allows us to obtain a large class of similar patterns. They seem to be of interest for the icosahedral quasicrystal modelling, but following the suggestions of the Janot–de Boissieu model, a better choice seems to be the patterns defined by the family of three-shell Y-clusters $Y\{(\alpha,\alpha,\alpha),(1,0,\tau),(\beta,0,0)\}$ (a dodecahedron, an icosahedron and an icosidodecahedron) depending on two real positive parameters. It is not the purpose of this paper to analyse these concrete patterns.

A real quasicrystal contains several sorts of atoms and it is very difficult to obtain a mathematical model describing the atomic positions. We think that our results offer the mathematical basis for a possible new approach to this open problem.

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